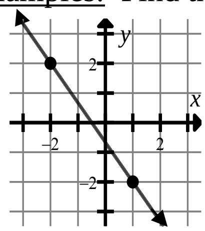
AP Calculus-AB **Name: ________________________________** **Per: _____**
Unit 0 Notes

LESSON 0.1 SLOPES, LINES, and CALCULATOR REVIEW

The slope of a line is symbolized by the letter "*m*".

Slope $= m = \dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$

Examples: Find the slopes of the lines containing each pair of points.

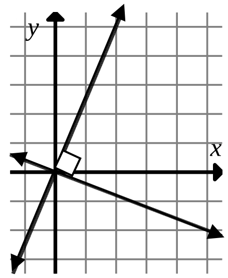
1. 

2. $(-2,0)$ *and* $(4,2)$

3. $(3,2)$ *and* $(2,2)$

4. $(3,2)$ *and* $(3,5)$

Parallel lines have equal slopes ($m_1 = m_2$).
Perpendicular lines have slopes which are opposite reciprocals $\left(m_1 = -\dfrac{1}{m_2}\right)$.



Equations for lines

- point-slope form: $y - y_1 = m(x - x_1)$
- slope-intercept form: $y = mx + b$ (where b is the *y*-intercept)
- general form : $Ax + By + C = 0$ (where A, B, and C are integers)

Examples: Find an equation of each line described.

5. a line through (2,3) with slope $m = -3$

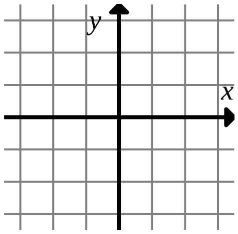
6. a vertical line through $(-1,2)$

7. a line through $(-1,2)$ parallel to the graph of $2x - 5y = 5$ (in slope-intercept form)

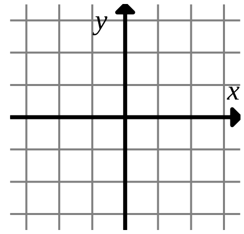
8. a line through $(-1,2)$ perpendicular to the graph of $2x - 5y = 5$ (in general form)

Examples: Draw a graph of each line.

9. $2x+3y=9$



10. $y=2$



Calculator Examples:

11. Find a window to show a complete graph of $y=f(x)=-0.2x^3-2.2x^2+1.6x+1$.
Indicate the scale on the graph or give your window setting.

12. Find the zeros of $y=f(x)=-0.2x^3-2.2x^2+1.6x+1$.

13. Find the points of intersection of $y=-x^3+12x^2+9x-3$ and $3x-y+5=0$. Write the equation you are solving.

14. Use a calculator to solve $|x^2-5|\geq 4$. Write your answer in both inequality notation and interval notation.

LESSON 0.2 FUNCTIONS, INVERSES, GRAPHING ADJUSTMENTS

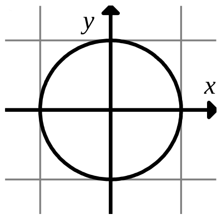
Relation: any set of ordered pairs (any set of points on a graph)

Function: a special type of relation. y is a function of x if for each x-value there is only one y-value. The graph of a function passes the vertical line test. This is written $y = f(x)$.

Domain: the set of all x-values
Range: the set of all y-values
} assuming y is a function of x

Examples: Determine whether each is a function of x.

1. $x+y=1$
2. $x^2+y^2=1$
3. $y=-x^2+1$
4. $x+y^2=1$

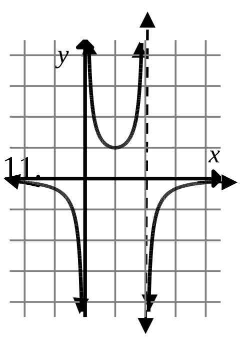


Given: $f(x)=3x-1$ and $g(x)=x^2$. Find the following.

5. $f(10) =$
6. $g(x+\Delta x)=$
7. $g(f(x))=$
8. $(f \circ g)(x)=$

Determine the domain and range for each function.

9. $f(x)=\sqrt{x-1}$
10. $g(x)=\dfrac{1}{x-2}$
11.



Do:
Ra:

Do:
Ra:

Do: Ra:

One-to-one Function: a function in which not only is there only one y for each x, but there is also only one x for each y. The graph passes the horizontal line test as well as the vertical line test.

Inverse Function: found by switching x and y and solving for the new y. $f^{-1}(x)$ is the symbol for the inverse of $f(x)$. Only one-to-one functions have inverse functions. Since x and y are switched to produce inverse functions, the domain of f is the range of f^{-1} and vice versa. If (a,b) is in the f function, then (b,a) is in the f^{-1} function.

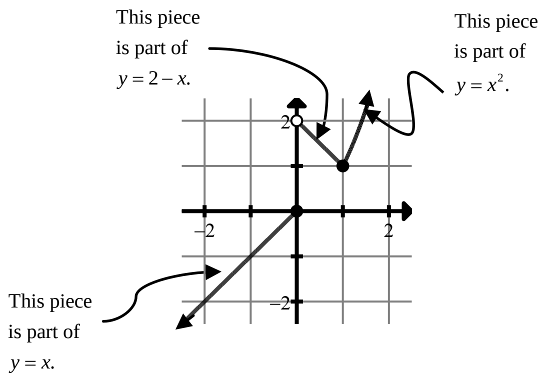
Examples:

12. Which of the relations in Examples 1-4 above is a function with an inverse function?

13. Find the inverse of $f(x)=2x^3-1$.

Piecewise Function: a function defined differently on different pieces of its domain.

Example: $f(x)=\begin{cases} x, & x\le 0 \\ 2-x, & 0<x<1 \\ x^2, & x\ge 1 \end{cases}$



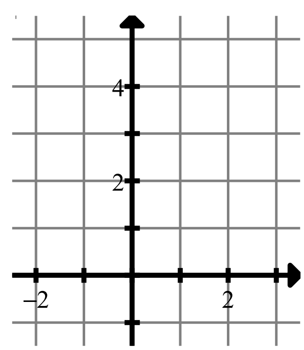
Examples:

14. Graph this piecewise function and give the domain and range.

$f(x)=\begin{cases} |x|, & x<1 \\ x+2, & x\ge 1 \end{cases}$

Do:

Ra:



Zeros: *x*-values for which y equals zero.
Conventionally, zeros are written as single values (e.g. $x = 2$ or $x = 5$) while *x*-intercepts are written as ordered pairs (e.g. (2,0) or (5,0)).

Find the zeros without using a calculator.

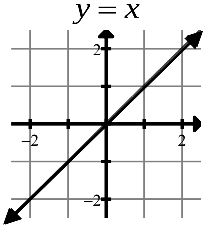
15. $f(x) = x^2 - 3x - 4$

16. $y = \dfrac{x^2 - 4}{x^2 + 4}$

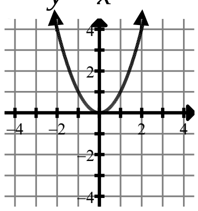
Parent Graphs

These graphs occur so frequently in this course that it would be worth your time to learn (memorize) them.

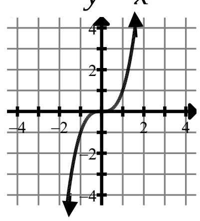
$y = x$



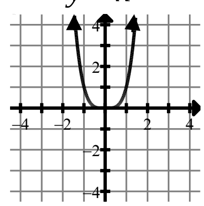
$y = x^2$



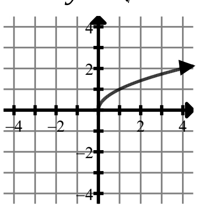
$y = x^3$



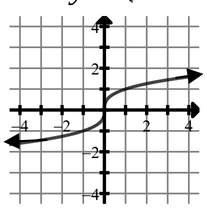
$y = x^4$



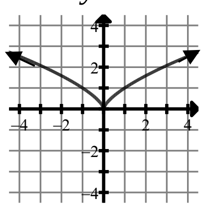
$y = \sqrt{x}$



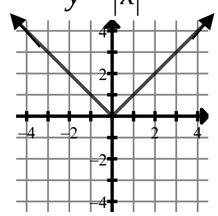
$y = \sqrt[3]{x}$



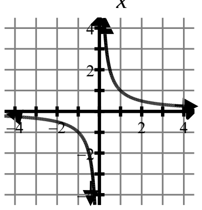
$y = x^{\frac{2}{3}}$



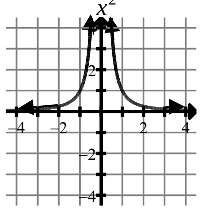
$y = |x|$



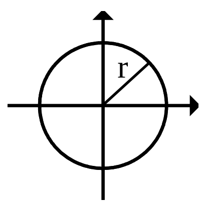
$y = \dfrac{1}{x}$



$y = \dfrac{1}{x^2}$



$x^2 + y^2 = r^2$

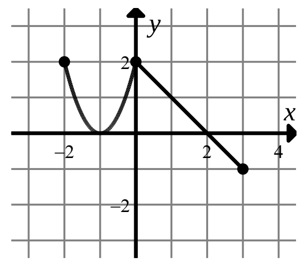


Graphing Adjustments to $y = f(x)$

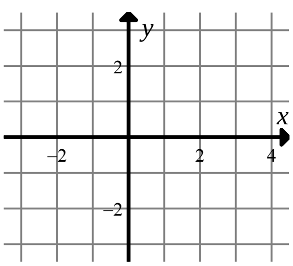
1. $y = -f(x)$ reflect across the x-axis
2. $y = f(-x)$ reflect across the y-axis
3. $y = f(x) + d$ shift up if $d > 0$, shift down if $d < 0$
4. $y = f(x + c)$ shift left if $c > 0$, shift right if $c < 0$
5. $y = a \cdot f(x)$ vertical stretch if $a > 1$, vertical squeeze if $a < 1$
   (assumes a is positive, if a is negative a reflection is needed)
6. $y = f(b \cdot x)$ horizontal squeeze if $b > 1$, horizontal stretch if $b < 1$
   (assumes b is positive, if b is negative a reflection is needed)
7. $y = |f(x)|$ reflect all points below the x-axis across the x-axis. Leave points above the x-axis alone.
8. $y = f(|x|)$ eliminate completely all points left of the y-axis. Leave points right of the y-axis alone. Replace the left half of the graph with a reflection of the right half. Your graph should then show y-axis symmetry.

Note: Adjustments to functions always produce functions.

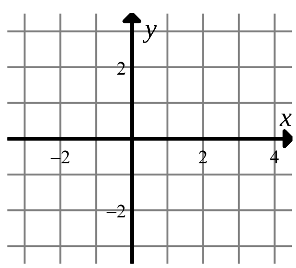
Examples: Use the graph of $y = f(x)$ shown to sketch the following:



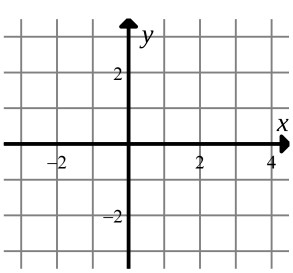
17. $y = f(x + 2)$



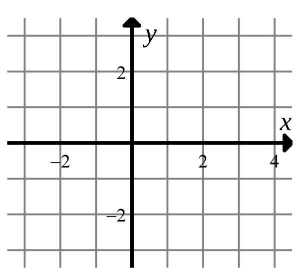
18. $y = -f(x) + 2$



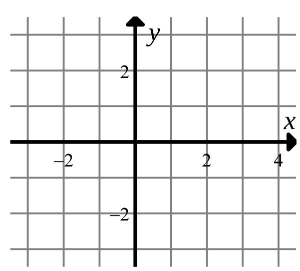
19. $y = \frac{1}{2} f(-x)$



20. $y = |f(2x)|$



21. $y = f(|x|)$



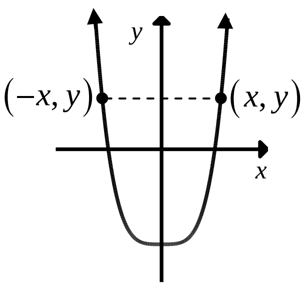
LESSON 0.3 INTERCEPTS, SYMMETRY, EVEN/ODD, INTERSECTIONS

x - and y - intercepts

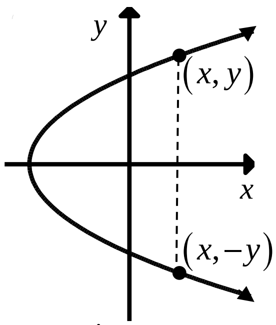
x-intercepts are points where a graph crosses or touches the x-axis. The y-coordinate is zero.
To find the x-intercept, let $y = 0$ and solve for x.
y-intercepts are points where a graph crosses or touches the y-axis. The x-coordinate is zero.
To find the y-intercept, let $x = 0$ and solve for y.

Example 1.
Find the x- and y-intercepts for $y^2 - 3 = x$.

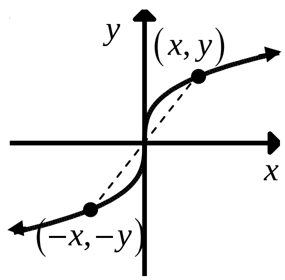
Symmetry



y-axis symmetry
reflection across
the y-axis



x-axis symmetry
reflection across
the x-axis



origin symmetry
reflection through
the origin (0,0)

Graphs can be symmetric to other lines and points. However, we will concentrate on these three.

Formal tests for symmetry:
1. y-axis: replacing x with $-x$ produces an equivalent equation
2. x-axis: replacing y with $-y$ produces an equivalent equation
3. origin: replacing x with $-x$ and y with $-y$ produces an equivalent equation

Informal tests for symmetry:
1. y-axis: substituting a number and its opposite for x give the same y-value
2. x-axis: substituting a number and its opposite for y give the same x-value
3. origin: substituting a number and its opposite for x give opposite y-values

Note: These informal tests are not foolproof. Think about whether other numbers would work the same. If your substitution produces zero, try another number.

Examples: Find the type(s) of symmetry for the graph of:

2. $y = 2x^3 - x$

3. $y = |x| - 2$

4. $|y| = x - 2$

Even/Odd Functions

A function is defined to be **even** if $f(-x) = f(x)$ for all x in the domain of $f(x)$. Even functions have graphs with *y*-axis symmetry. Examples: $y = x^2$, $y = x^4$, $y = x^2 + 3$, $y = x^4 + x^2$

A function is defined to be **odd** if $f(-x) = -f(x)$ for all x in the domain of $f(x)$. Odd functions have graphs with origin symmetry. Examples: $y = x$, $y = x^3$, $y = x^5$, $y = x^5 - x^3$

Examples: Determine whether the following functions are even, odd, or neither.

5. $f(x) = x^3 - x$

6. $g(x) = x^2 - 4$

7. $h(x) = x^2 + 2x + 2$

Points of Intersection of Two Graphs (without a calculator)

Method 1. Solve one equation for one variable and substitute into the other equation.

Method 2. Solve both equations for the same variable and set equal.

Example 8. Without using a calculator, find all points of intersection for the graphs of $x - y = 1$ and $x^2 - y = 3$.

LESSON 0.4 REVIEW OF BASIC TRIGONOMETRY

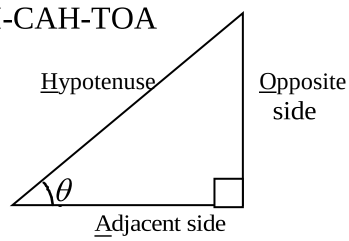
Basic Right Triangle Trigonometry:

The basic right triangle trigonometric ratios are given by SOH-CAH-TOA

$\sin\theta = \dfrac{opp}{hyp}$ (SOH) $\csc\theta = \dfrac{1}{\sin\theta}$

$\cos\theta = \dfrac{adj}{hyp}$ (CAH) $\sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{opp}{adj}$ (TOA) $\cot\theta = \dfrac{1}{\tan\theta}$

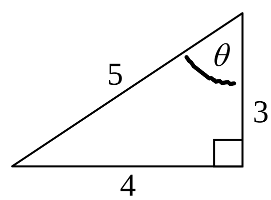


$0^\circ < \theta < 90^\circ$

When using right triangle trigonometry, angles are usually measured in degrees.

Example 1: Use the triangle at right to find

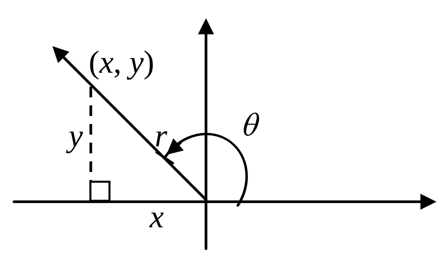
a. $\sin\theta$ b. $\cos\theta$ c. $\tan\theta$ d. $\sec\theta$



Trigonometric Functions Defined as Circular Functions:

Angles in a right triangle must be positive and less than or equal to 90°. A less restrictive way of defining trigonometric (trig) ratios is to use angles which can be any measure.

At right is an angle in standard position. The vertex of the angle is the origin. The initial side of the angle is the positive *x*-axis. In the figure shown, the terminal side was formed by a counter-clockwise rotation, so the measure of the angle, (θ), is positive Clockwise rotations produce negative angles.



When trig functions are defined using rotations from an initial ray (side) in the coordinate plane, they are called circular functions. In Calculus, angles are usually defined by circular trig functions and are almost always measured in radians. $(2\pi^{R} = 360^{O})$

The circular function trig definitions are (see figure):

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

$$\begin{cases} \theta \text{ is any measure} \\ r = \sqrt{x^2 + y^2} \quad \text{(Positive)} \\ x \text{ and } y \text{ may be } +, -, \text{ or } 0 \end{cases}$$

Example 2: Find $\sin\theta$, $\csc\theta$, and $\cot\theta$, if θ is an angle in standard position whose terminal side passes through the point $(-5, 2)$.

Circular function trigonometry makes use of reference angles in triangles and is really not much different than right triangle trigonometry. Think of it as an extension of right triangle trig.

$30^{O} - 60^{O} - 90^{O}$ and $45^{O} - 45^{O} - 90^{O}$ reference triangles can be used to find trig ratios of angles which are multiples of 30^{O} or 45^{O}.

Example 3: Draw angles in standard position and make "reference triangles" to find:

a. $\cos 210^{O}$ b. $\tan 315^{O}$

Example 4: Since 2π radians $= 360^{O}$, it follows that $\pi^{R} = 180^{O}$, and the following common radian measures should be easy to think about in degrees. Convert each common radian measure to degrees.

a. $\frac{\pi}{2} =$ b. $\frac{\pi}{4} =$ c. $\frac{\pi}{3} =$ d. $\frac{\pi}{6} =$

Example 5: Convert from radians to degrees or degrees to radians without using a calculator.

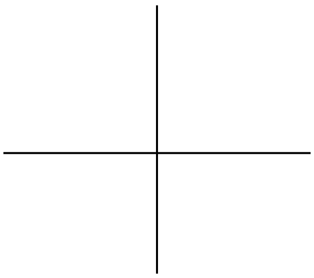
a. $\frac{5\pi}{4} =$

b. $270^{o} =$

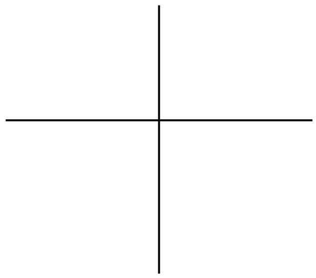
c. $-120^{o} =$

Examples: Draw angles in standard position, and make "reference triangles" to find the following without using a calculator:

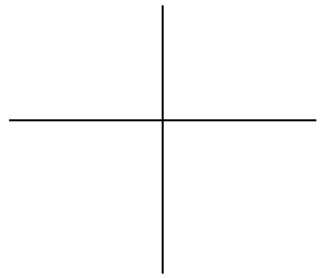
6. $\tan\frac{5\pi}{6}$



7. $\cos\left(\frac{-3\pi}{4}\right)$



8. $\csc\frac{5\pi}{3}$



A unit circle is created by letting $r = 1$ when dealing with the circular trig functions.

Then, $\sin\theta = y$, $\cos\theta = x$, and $\tan\theta = \frac{y}{x}$.

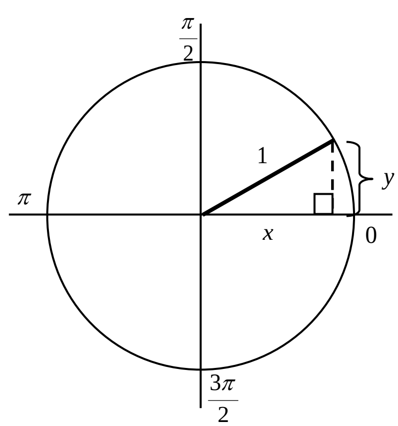
Example 9: Use a unit circle to find:

a. $\sin\frac{\pi}{6}$ b. $\sin 0$ c. $\cos 0$

d. $\sin\frac{\pi}{2}$ e. $\cos\frac{\pi}{2}$ f. $\sin\pi$

g. $\tan\pi$ h. $\sin\frac{3\pi}{2}$ i. $\cos\frac{3\pi}{2}$

j. $\cos(-\pi)$ k. $\tan\left(\frac{-\pi}{2}\right)$



As you can see from Example 9, the unit circle is particularly useful when finding trig ratios for the quadrant separators (since no "reference triangles" can be built for them).

Sine and cosine are the two most important trig functions. The other trig functions can all be built as ratios of the sine and cosine functions.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$ $\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$

Example 10: If $\sin\theta = \frac{3}{5}$, find the possible values for

a. $\csc\theta$ b. $\cos\theta$ c. $\tan\theta$

Solving trigonometric equations requires you to "**work backwards**" from ratios to angles.

Example 11: Solve the following trig equations without using a calculator. Find all of the solutions in the interval $\left[0, 2\pi\right)$.

a. $\csc x = \frac{-2}{\sqrt{3}}$ b. $\cot\theta = \sqrt{3}$

c. $2\cos^2\theta - 1 = 0$ **Note**: $\cos^2\theta$ means $(\cos\theta)^2$. This is trig symbolism.

For these problems, you must be very careful with your "**SIGNS**."

LESSON 0.5 TRIGONOMETRY WITH A CALCULATOR, GRAPHS OF TRIGONOMETRIC FUNCTIONS

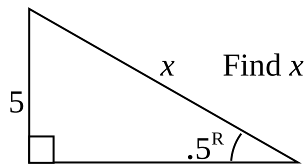
When using a calculator with trig functions, it is important that the calculator is set in the correct mode (radians or degrees). In Calculus, we will deal almost entirely with radian measure.

Example 1: Use a calculator to find:

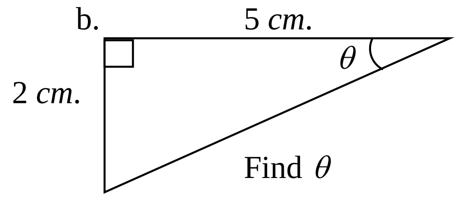
a. $\sin 2$ b. $\tan\left(\frac{-\pi}{5}\right)$ c. $\sec 1.3$

Example 2: Use a calculator to find the missing measure in each triangle.

a.



b.



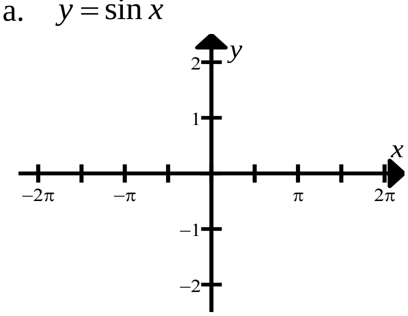
Graphs of Trig Functions:

Trig functions are periodic (their graphs repeat after a certain period or cycle).
The sine, cosine, cosecant, and secant functions all have a period of 2π.
The tangent and cotangent functions have a period of π.

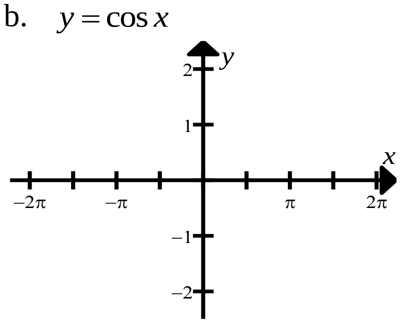
You should be able to easily graph the trig functions by using trig values at $x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$, and by using the fact that the functions are periodic.
(You should also use $x = \pm\frac{\pi}{4}$ for the tangent and cotangent graphs.)

Example 3: Graph each of the following.

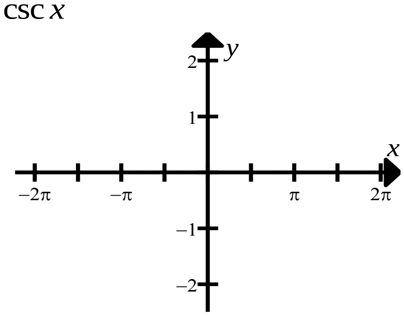
a. $y = \sin x$



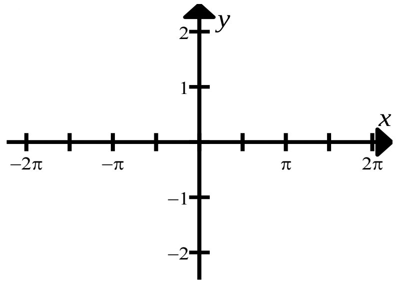
b. $y = \cos x$



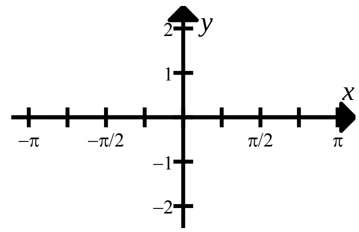
c. $y = \csc x$



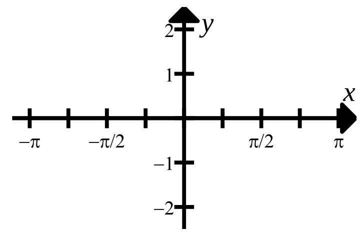
d. $y = \sec x$



e. $y = \tan x$



f. $y = \cot x$



Remember: Each of these last two functions has a period of π.

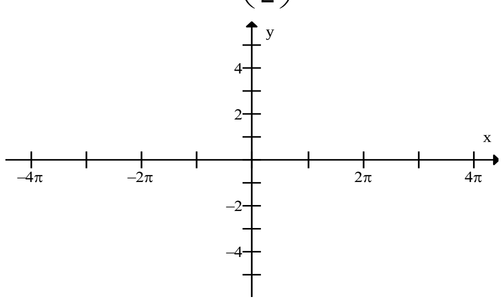
You should be able to use the parent trig graphs to graph functions of the form $y = a\sin\big(b(x+c)\big)+d$.
The chart below provides an aid, but remember to think of "adjustments to graphs."

FUNCTION	**PERIOD**	**AMPLITUDE**	**HORIZONTAL SHIFT**	**VERTICAL SHIFT**
$y = a\ \sin\big(b(x+c)\big)+d$ $y = a\ \cos\big(b(x+c)\big)+d$	$\frac{2\pi}{\lvert b\rvert}$	$\lvert a\rvert$	$-c$	d
$y = a\ \tan\big(b(x+c)\big)+d$ $y = a\ \cot\big(b(x+c)\big)+d$	$\frac{\pi}{\lvert b\rvert}$	None	$-c$	d
$y = a\ \sec\big(b(x+c)\big)+d$ $y = a\ \csc\big(b(x+c)\big)+d$	$\frac{2\pi}{\lvert b\rvert}$	None	$-c$	d

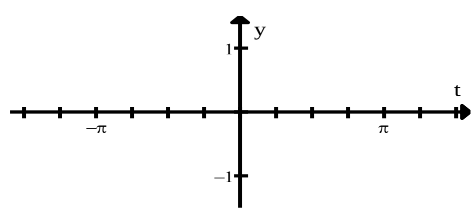
When c is positive, the horizontal shift is to the left. When c is negative, the horizontal shift is to the right. Horizontal shift is often called phase shift for periodic functions.

Example 4: Without using a calculator, sketch two cycles of:

a. $f(x) = -5\cos\left(\frac{x}{2}\right)$



b. $g(t) = \sin\left(2t - \frac{\pi}{2}\right)$



The sine and cosine functions are related to each other by the basic Pythagorean Identity:

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{or} \quad \sin^2 x + \cos^2 x = 1$$

Example 5: Use the Pythagorean Identity to rewrite $2\cos\theta - \sin^2\theta = -2$ in a form which only contains one trig function. Then, without using a calculator, solve for θ on the interval $[0, 2\pi)$.

Use your calculator to verify your solution.

LESSON 0.6 EXPONENTIAL FUNCTIONS

An **exponential function** is a function represented by a constant base with a variable exponent. For example, $f(x) = 2^x$, $y = e^x$, and $g(x) = 3^{x^2-5}$ are exponential functions.
These basic properties of exponents are used when working with exponential functions.

For a and b positive real numbers and x and y any real numbers:		
1. $a^0 = 1$	2. $a^x a^y = a^{x+y}$	3. $\dfrac{a^x}{a^y} = a^{x-y}$
4. $(a^x)^y = a^{xy}$	5. $(ab)^x = a^x b^x$	6. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
7. $a^{-x} = \dfrac{1}{a^x}$	Note: $(a+b)^x \neq a^x + b^x$	

When simplifying, do not leave answers with negative exponents.

Examples: Simplify without using a calculator.

1. $27^{\frac{4}{3}}$

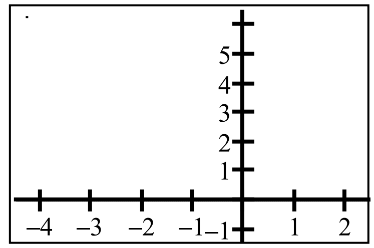
2. $\left(e + \dfrac{1}{e}\right)^0$

3. $\left(\dfrac{e^5 \cdot e^{-3}}{e^4}\right)^2$

4. $5^3 \cdot 25^{-2}$

5. Solve $9^x = 27$ without using a calculator.

6. Use a calculator to carefully graph $y = 2^x$, $y = 5^x$, and $y = e^x$ in the same coordinate plane. Do you see any similarities in the graphs?



Graphs of Exponential Functions: If $f(x) = a^x$ and $a > 1$, then

1. The domain of $f(x)$ is $(-\infty, \infty)$. The range of $f(x)$ is $(0, \infty)$.

2. The graph of $f(x)$ is continuous, increasing, concave upward, and one-to-one (has an inverse function).

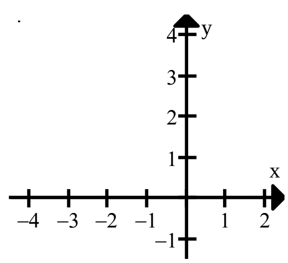
3. The x-axis is a <u>horizontal asymptote</u> to the left: $\lim_{x\to-\infty} f(x) = 0$. *
(Also, $\lim_{x\to\infty} f(x) = \infty$) *

4. The y-intercept is (0, 1). Another key point is (1, a).

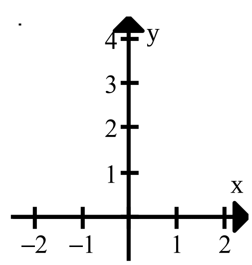
*This notation using limits will be developed completely in the next unit.

The letter e used as a base in Examples 2, 3, and 6, is not an unknown. It is a number called the <u>natural base</u> for exponential functions. It is the most common base in Calculus, because functions with base e are easier to differentiate and integrate than functions with other bases. By definition, $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$. To three decimal places, $e \approx 2.718$.

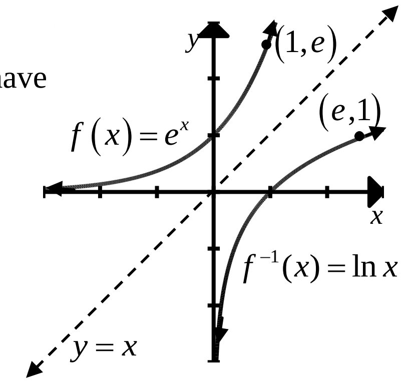
<u>Example 7</u>: Without using a calculator, sketch a graph of $y = e^x$.



<u>Example 8</u>: Using adjustments to the graph from Example 7, graph $f(x) = e^{-x} + 1$ without using a calculator. Write an equation for the graph's asymptote.



LESSON 0.7 LOGARITHMIC FUNCTIONS



Since $f(x)=e^{x}$ is one-to-one (continuous and increasing), it must have an inverse. However, if you switch x and y in the equation $y=e^{x}$ to get $x=e^{y}$, you cannot isolate the new y by using algebraic methods. So, we must define $f^{-1}(x)$ for the function $f(x)=e^{x}$.
For $f(x)=e^{x}$, $f^{-1}(x)$ is called the **natural logarithmic function**, and we write $f^{-1}(x)=\ln x$ (so that $x=e^{y}$ and $y=\ln x$ must be equivalent). In general, if $f(x)=a^{x}$ $(a>0)$, then $f^{-1}(x)=\log_{a} x$ (so that $x=a^{y}$ and $y=\log_{a} x$ must be equivalent).

Note: $\log_{e} x$ is usually written as $\ln x$ and $\log_{10} x$ is usually written simply as $\log x$.

Graphs of Logarithmic Functions: If $f(x)=\log_{a} x$ and $a>1$, then

1. The domain of $f(x)$ is $(0,\infty)$.
   The range of $f(x)$ is $(-\infty,\infty)$.

2. The graph of $f(x)$ is continuous, increasing, concave downward, and one-to-one (has an inverse function).

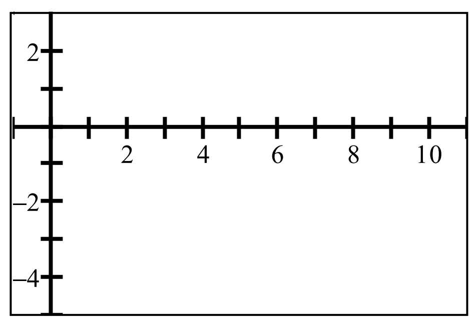
3. The y-axis is a vertical asymptote downward: $\lim_{x\to 0} f(x)=-\infty$ *
   (Also, $\lim_{x\to\infty} f(x)=\infty$) *

4. The x-intercept is (1, 0).
   Another key point is $(a, 1)$.

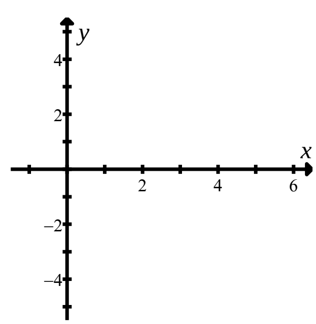
*This notation using limits will be developed completely in the next unit.

Compare these graphical characteristics of $f(x)=\log_{a} x$ to those of $f(x)=a^{x}$ from Lesson 0.6 (page 22).

Example 1: Use a calculator to graph $y=\ln x$ and $y=\log x$ in the same coordinate plane. Do you see any similarities in the graphs?



Example 2: Without using a calculator, sketch a graph of $y=|\ln(x-2)|$. Write an equation for the graph's asymptote.



For changing forms of an equation involving exponentials or logarithms, we use the following **Change of Form Definition**:

Exponential form	$\left\{\begin{array}{l} x = e^y \leftrightarrow y = \ln x \\ x = a^y \leftrightarrow y = \log_a x \end{array}\right\}$	Logarithmic form

Example 3: Change the following equations from exponential form to logarithmic form or vice versa.

a. $3^4 = 81$ b. $e^0 = 1$ c. $\log(.1) = -1$

Example 4:

a. Since $e^0 = 1$, $\ln 1 =$ b. Since $e^1 = e$, $\ln e =$

c. Because the natural exponential function and the natural logarithmic function are inverses,

$\ln e^n = e^{\ln n} =$

Example 5: Use the inverse idea from Example 4c. to simplify.

a. $\ln e^{\sqrt{2}} =$ b. $e^{\ln(3x)} =$ c. $10^{\log 2} =$ d. $\log_2 2^{x^2} =$

Properties of Logarithms:

1. $\ln(ab) = \ln a + \ln b$
2. $\ln \frac{a}{b} = \ln a - \ln b$
3. $\ln a^n = n \ln a$

These properties work for any bases, but only if $a > 0$ and $b > 0$

Example 6: Expand using Logarithm Properties 1-3 above.

a. $\ln \frac{5}{8}$ b. $\ln \sqrt[3]{x^2 + 1}$

Example 7: Condense into a single logarithm. $(x > 0$ and $y > 0)$

a. $-3\ln x + 5\ln y$

b. $\frac{1}{2}\ln x + \ln(x+1) - 3\ln y$

Example 8: Solve for *x*.

a. $y = e^{2x-5} + 6$

b. $\log_2 x - \log_2(x-8) = 3$

Change of Base Formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$

Since the only two logarithmic bases on your calculator are 10 (log key) and e ($\ln$ key), you will change bases on your calculator in one of two ways:

$$\log_a x = \frac{\log x}{\log a} \qquad \text{or} \qquad \log_a x = \frac{\ln x}{\ln a}$$

Example 9: Use your calculator to find $\log_7 112$ to 3 or more decimal places.

Example 10:

a. Find an exact value for x, if $3^{x+2} = 6$.

b. Use your calculator to find a decimal value for your answer from Part a. to 3 or more decimal places.

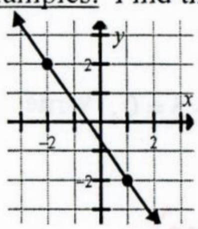
Lesson 0.1 **SLOPES, LINES, CALCULATOR REVIEW**

The slope of a line is symbolized by the letter "*m*".

Slope $= m = \dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$

Examples: Find the slopes of the lines containing each pair of points.

1.



$m = -\frac{4}{3}$

2. $(-2,0)$ *and* $(4,2)$

$m = \dfrac{2-0}{4-(-2)} = \dfrac{2}{6} = \dfrac{1}{3}$

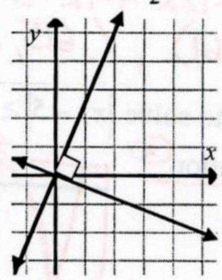
3. $(3,2)$ *and* $(2,2)$

$m = \dfrac{2-2}{2-3} = 0$

4. $(3,2)$ *and* $(3,5)$

$m = \dfrac{5-2}{3-3}$ undefined

Parallel lines have equal slopes $(m_1 = m_2)$.
Perpendicular lines have slopes which are opposite reciprocals $\left(m_1 = -\dfrac{1}{m_2}\right)$.



Equations for lines

point-slope form: $y - y_1 = m(x - x_1)$

slope-intercept form: $y = mx + b$ (where b is the y-intercept)

general form : $Ax + By + C = 0$ (where A, B, and C are integers)

Examples: Find an equation of each line described.

5. a line through (2,3) with slope $m = -3$

$y - 3 = -3(x - 2)$

6. a vertical line through $(-1,2)$

$x = -1$

7. a line through $(-1,2)$ parallel to the graph of $2x - 5y = 5$ (in slope-intercept form)

① $-5y = 5 - 2x$
$y = -1 + \frac{2}{5}x$
$m_1 = \frac{2}{5}$

② $m_2 = \frac{2}{5}$
$y - 2 = \frac{2}{5}(x + 1)$

③ $y = \frac{2}{5}x + \frac{2}{5} + 2$
or $y = \frac{2}{5}x + 2\frac{2}{5}$

8. a line through $(-1,2)$ perpendicular to the graph of $2x - 5y = 5$ (in general form)

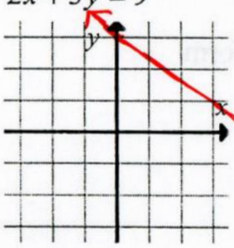
① $m_1 = \frac{2}{5}$
$m_2 = -\frac{5}{2}$

② $y - 2 = -\frac{5}{2}(x + 1)$

③ $y - 2 = -\frac{5}{2}x - \frac{5}{2}$ ↓ Mult. by 2
$2y - 4 = -5x - 5$
$5x + 2y + 1 = 0$
or $-5x - 2y - 1 = 0$

Examples: Draw a graph of each line.

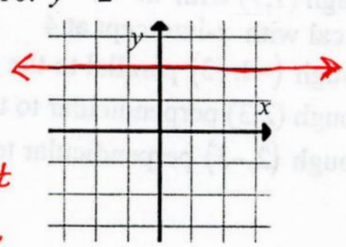
9. $2x + 3y = 9$



METHOD 1
(Find 2 intercepts)
$0 + 3y = 9$ $\quad 2x + 0 = 9$
$y = 3$ $\quad x = 4.5$

METHOD 2
(Put into slope-intercept form)
$3y = -2x + 9$ $\quad m = -\frac{2}{3}$
$y = -\frac{2}{3}x + 3$ $\quad b = 3$

10. $y = 2$

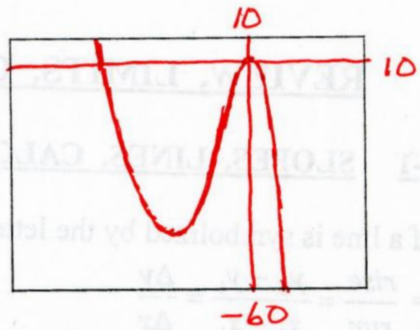


horizontal line
$y =$ ___

Calculator Examples:

11. Find a window to show a complete graph of $y = f(x) = -0.2x^3 - 2.2x^2 + 1.6x + 1$.

    Indicate the scale on the graph or give your window setting.



$[-20, 10]$ by $[-60, 10]$
x y

12. Find the zeros of $y = f(x) = -0.2x^3 - 2.2x^2 + 1.6x + 1$.

$f(x) = 0$

$x = -11.649$ or -11.650, $x = -.406$, $x = 1.056$

13. Find the points of intersection of $y = -x^3 + 12x^2 + 9x - 3$ and $3x - y + 5 = 0$. Write the equation you are solving. $3x - y + 5 = 0 \rightarrow y = 3x + 5$

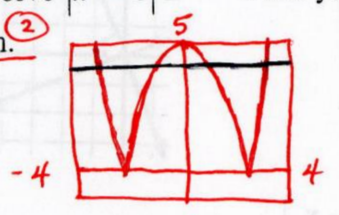
Equation: $-x^3 + 12x^2 + 9x - 3 = 3x + 5$

$(-1.046, 1.861)$, $(.615, 6.845)$, $(12.430, 42.292)$

14. Use a calculator to solve $|x^2 - 5| \geq 4$. Write your answer in both inequality notation ① and interval notation. ②

$y_1 = |x^2 - 5|$

$y_2 = 4$



Intersections at $x = -3, -1, 1, 3$

$y_1 \geq y_2$ for:

$x \leq -3$, $-1 \leq x \leq 1$, $x \geq 3$ ①

$(-\infty, -3]$, $[-1, 1]$, $[3, \infty)$ ②

LESSON 0.2 FUNCTIONS, INVERSES, GRAPHING ADJUSTMENTS

Relation: any set of ordered pairs (any set of points on a graph)

Function: a special type of relation. y is a function of x if for each x-value there is only one y-value. The graph of a function passes the vertical line test. This is written $y = f(x)$.

Domain: the set of all x-values
Range: the set of all y-values
} assuming y is a function of x

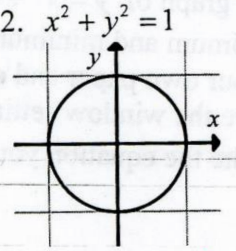
Examples: Determine whether each is a function of x.

1. $x + y = 1$

$y = 1 - x$

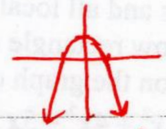
LINE
YES

2. $x^2 + y^2 = 1$



NO $y^2 = 1 - x^2 \rightarrow y = \pm\sqrt{1-x^2}$

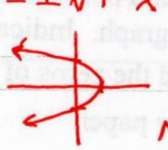
3. $y = -x^2 + 1$



PARABOLA
YES

4. $x + y^2 = 1$

$y^2 = 1 - x$
$y = \pm\sqrt{1-x}$



SIDEWAYS PARABOLA
NO

Given: $f(x) = 3x - 1$ and $g(x) = x^2$. Find the following.

5. $f(10) =$ $3(10) - 1$ or 29

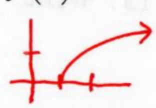
6. $g(x + \Delta x) =$ $(x + \Delta x)^2$ or $x^2 + 2x\Delta x + (\Delta x)^2$ (not suggested)

7. $g(f(x)) =$ $g(3x-1) = (3x-1)^2$ or $9x^2 - 6x + 1$

8. $(f \circ g)(x) =$ $f(g(x)) = f(x^2) = 3x^2 - 1$

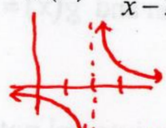
Determine the domain and range for each function.

9. $f(x) = \sqrt{x-1}$



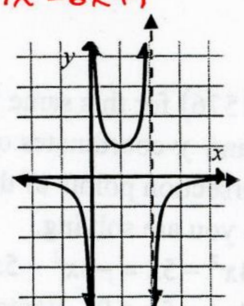
Do: $x \geq 1$
Ra: $y \geq 0$

10. $g(x) = \dfrac{1}{x-2}$



Do: $x \neq 2$
Ra: $y \neq 0$

11.



Do: $x \neq 0, 2$ Ra: $y < 0$, $y \geq 1$

One-to-one Function: a function in which not only is there only one y for each x, but there is also only one x for each y. The graph passes the horizontal line test as well as the vertical line test.

Inverse Function: found by switching x and y and solving for the new y. $f^{-1}(x)$ is the symbol for the inverse of $f(x)$. Only one-to-one functions have inverse functions. Since x and y are switched to produce inverse functions, the domain of f is the range of f^{-1} and vice versa. If (a,b) is in the f function, then (b,a) is in the f^{-1} function.

Examples:

12. Which of the relations in Examples 1-4 is a function with an inverse function? ONLY EX. 1

13. Find the inverse of $f(x) = 2x^3 - 1$. $y = 2x^3 - 1$ $y = \sqrt[3]{\frac{x+1}{2}} = f^{-1}(x)$

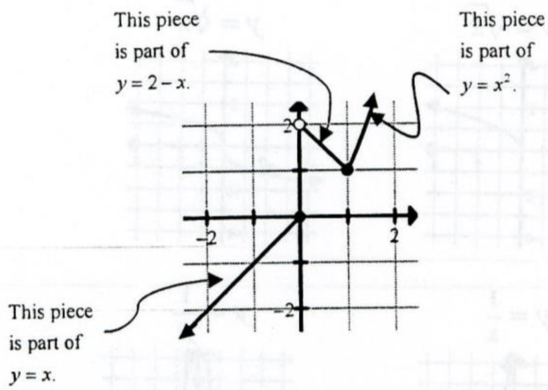
① Switch x and y ② Isolate the new y

$x = 2y^3 - 1$ $x + 1 = 2y^3$

$\frac{x+1}{2} = y^3$

Piecewise Function: a function defined differently on different pieces of its domain.

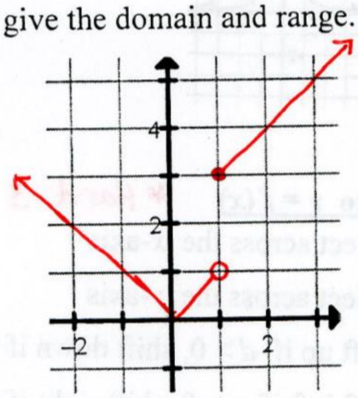
Example: $f(x) = \begin{cases} x, & x \le 0 \\ 2 - x, & 0 < x < 1 \\ x^2, & x \ge 1 \end{cases}$



Examples:

14. Graph this piecewise function and give the domain and range.

$f(x) = \begin{cases} |x|, & x < 1 \\ x + 2, & x \ge 1 \end{cases}$



Do: all reals

Ra: $y \ge 0$ or $f(x) \ge 0$

Zeros: x-values for which y equals zero.
Conventionally, zeros are written as single values (e.g. $x = 2$ or $x = 5$) while x-intercepts are written as ordered pairs (e.g. (2,0) or (5,0)).

Find the zeros without using a calculator. Use algebraic techniques.

15. $f(x) = x^2 - 3x - 4$

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4, -1$

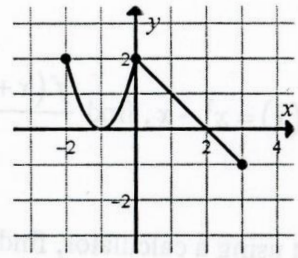
16. $y = \frac{x^2 - 4}{x^2 + 4}$

$x^2 - 4 = 0$ NOT $x^2 + 4 = 0$

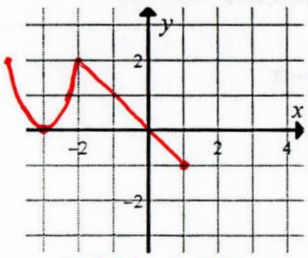
$x^2 = 4$

$x = \pm 2$

Examples: Use the graph of $y = f(x)$ shown to sketch the following:

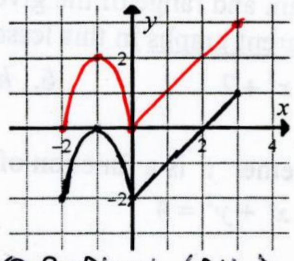


17. $y = f(x+2)$



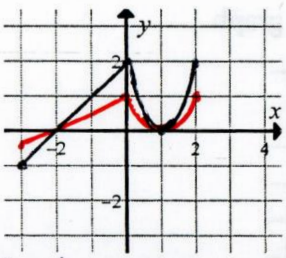
Shift left 2

18. $y = -f(x)+2$



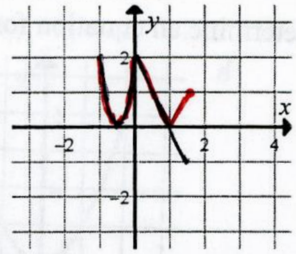
① Reflect (flip) across x-axis: $y=-f(x)$
② Shift up 2: $y=-f(x)+2$

19. $y = \frac{1}{2} f(-x)$



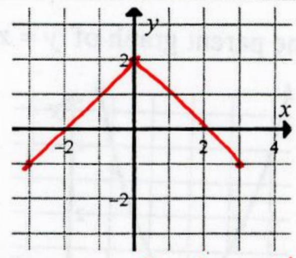
① Reflect across y-axis $y=f(-x)$
② Squeeze vertically by a factor of $\frac{1}{2}$. (Multiply y-values by $\frac{1}{2}$) $y=\frac{1}{2}f(-x)$

20. $y = |f(2x)|$



① Squeeze horizontally by a factor of $\frac{1}{2}$. (multiply x-values by $\frac{1}{2}$) $y=f(2x)$
② Reflect the points lying below the x-axis across the x-axis. $y=|f(2x)|$

21. $y = f(|x|)$



① Eliminate all points to the left of the y-axis.
② Replace the left half of the graph with the mirror image (reflection) of the right half. You should see symmetry to the y-axis.

LESSON 0.3 INTERCEPTS, SYMMETRY, EVEN/ODD, INTERSECTIONS

x- and y-intercepts

x-intercepts are points where a graph crosses or touches the x-axis. The y-coordinate is zero. To find the x-intercept, let $y = 0$ and solve for x.

y-intercepts are points where a graph crosses or touches the y-axis. The x-coordinate is zero. To find the y-intercept, let $x = 0$ and solve for y.

Example 1.

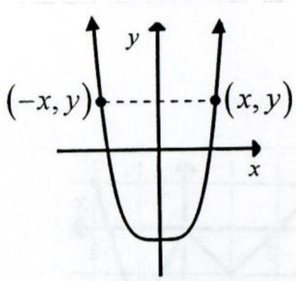
Find the x- and y-intercepts for $y^2 - 3 = x$.

X-int: $0 - 3 = x$ y-int: $y^2 - 3 = 0$ $y^2 = 3$

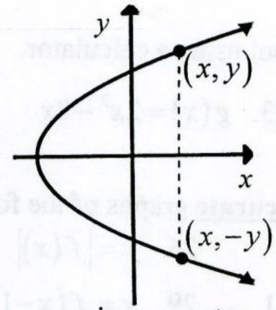
Thought: Let $y=0$ $(-3,0)$ Thought: Let $x=0$ $y = \pm\sqrt{3}$

$(0, \pm\sqrt{3})$

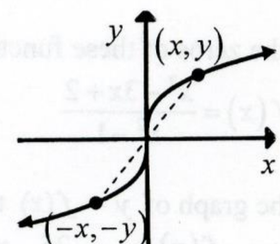
Symmetry



y-axis symmetry
reflection across the y-axis



x-axis symmetry
reflection across the x-axis



origin symmetry
reflection through the origin (0,0)

Graphs can be symmetric to other lines and points. However, we will concentrate on these three.

Formal tests for symmetry:
1. y-axis: replacing x with $-x$ produces an equivalent equation
2. x-axis: replacing y with $-y$ produces an equivalent equation
3. origin: replacing x with $-x$ and y with $-y$ produces an equivalent equation

Informal tests for symmetry:
1. y-axis: substituting a number and its opposite for x give the same y-value
2. x-axis: substituting a number and its opposite for y give the same x-value
3. origin: substituting a number and its opposite for x give opposite y-values

Note: These informal tests are not foolproof. Think about whether other numbers would work the same. If your substitution produces zero, try another number.

Examples: Find the type(s) of symmetry for the graph of:

2. $y = 2x^3 - x$ 3. $y = |x| - 2$ 4. $|y| = x - 2$

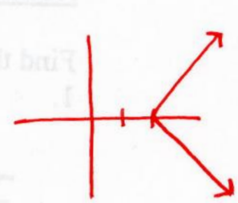
Informal: If $x=1$, $y=1$; If $x=-1$, $y=-1$ | If $x=1$, $y=-1$; If $x=-1$, $y=-1$ | If $x=3$, $y=\pm 1$ (opposite y-values produce the same x-value)

Formal: $2(-x)^3-(-x) = -2x^3+x = -y$ | or $|-x|-2 = |x|-2$

origin symmetry

or

y-axis symmetry



or graph

x-axis symmetry

NOT a function

Even/Odd Functions

A function is defined to be **even** if $f(-x) = f(x)$ for all x in the domain of $f(x)$. Even functions have graphs with y-axis symmetry. Examples: $y = x^2$, $y = x^4$, $y = x^2 + 3$, $y = x^4 + x^2$ How about $y = |x^3|$?

A function is defined to be **odd** if $f(-x) = -f(x)$ for all x in the domain of $f(x)$. Odd functions have graphs with origin symmetry. Examples: $y = x$, $y = x^3$, $y = x^5$, $y = x^5 - x^3$

Examples: Determine whether the following functions are even, odd, or neither.

5. $f(x) = x^3 - x$ 6. $g(x) = x^2 - 4$ 7. $h(x) = x^2 + 2x + 2$

$f(2) = 8-2 = 6$ | $g(1) = -3$ | $h(1) = 5$

$f(-2) = -8+2 = -6$ | $g(-1) = -3$ | $h(-1) = 1$

ODD (origin sym.) | EVEN (y-axis sym.) | NEITHER

Points of Intersection of Two Graphs (without a calculator)

Method 1. Solve one equation for one variable and substitute into the other equation.

Method 2. Solve both equations for the same variable and set equal.

Example 8. Without using a calculator, find all points of intersection for the graphs of $x - y = 1$ and $x^2 - y = 3$.

$y = x-1$ $y = x^2-3$

METHOD 1

Substitute:

$x^2-(x-1) = 3$

$x^2-x+1 = 3$

$x^2-x-2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

(2,1) (−1,−2)

METHOD 2

Since $y = x-1$ and $y = x^2-3$

$x-1 = x^2-3$

$0 = x^2-x-2$

$0 = (x-2)(x+1)$

$x = 2, -1$

(2,1) (−1,−2)

METHOD 3 (Elimination)

$-x+y = -1$

$x^2-y = 3$

$x^2-x = 2$

$x^2-x-2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

(2,1) (−1,−2)

LESSON 0.4 **REVIEW OF BASIC TRIGONOMETRY**

Basic Right Triangle Trigonometry:

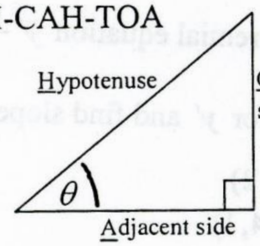
The basic right triangle trigonometric ratios are given by SOH-CAH-TOA

sine $\theta = \dfrac{opp}{hyp}$ (SOH) cosecant $\theta = \dfrac{1}{\sin\theta}$

cosine $\theta = \dfrac{adj}{hyp}$ (CAH) secant $\theta = \dfrac{1}{\cos\theta}$

tangent $\theta = \dfrac{opp}{adj}$ (TOA) cotangent $\theta = \dfrac{1}{\tan\theta}$

(cosecant, secant, cotangent: Reciprocal Functions)

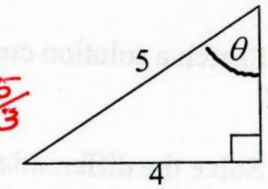


$0^\circ < \theta < 90^\circ$

When using right triangle trigonometry, angles are usually measured in degrees.

Example 1: Use the triangle at right to find

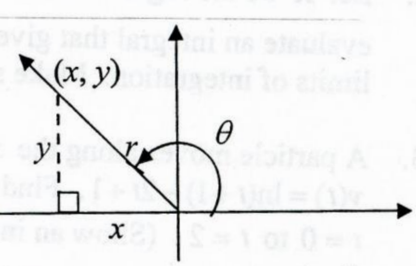
a. $\sin\theta = \frac{4}{5}$ b. $\cos\theta = \frac{3}{5}$ c. $\tan\theta = \frac{4}{3}$ d. $\sec\theta = \frac{5}{3}$



Trigonometric Functions Defined as Circular Functions:

Angles in a right triangle must be positive and less than or equal to 90°. A less restrictive way of defining trigonometric (trig) ratios is to use angles which can be any measure.

At right is an angle in standard position. The vertex of the angle is the origin. The initial side of the angle is the positive x-axis. In the figure shown, the terminal side was formed by a counter-clockwise rotation, so the measure of the angle, (θ), is positive. Clockwise rotations produce negative angles.



When trig functions are defined using rotations from an initial ray (side) in the coordinate plane, they are called circular functions. In Calculus, angles are usually defined by circular trig functions and are almost always measured in radians. $(2\pi^{R} = 360^\circ)$

The circular function trig definitions are (see figure):

$\sin\theta = \dfrac{y}{r}$ $\csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r}$ $\sec\theta = \dfrac{r}{x}$

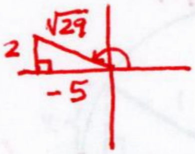
$\tan\theta = \dfrac{y}{x}$ $\cot\theta = \dfrac{x}{y}$

θ is any measure

$r = \sqrt{x^2 + y^2}$ (Positive)

x and y may be +, −, or 0

Example 2: Find $\sin\theta$, $\csc\theta$, and $\cot\theta$, if θ is an angle in standard position whose terminal side passes through the point $(-5,2)$.

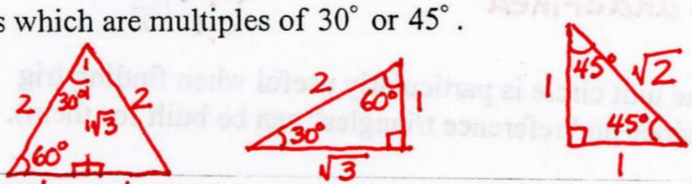


$r = \sqrt{25+4} = \sqrt{29}$ $\sin\theta = \frac{2}{\sqrt{29}}$ $\cot\theta = -\frac{5}{2}$

$\csc\theta = \frac{\sqrt{29}}{2}$

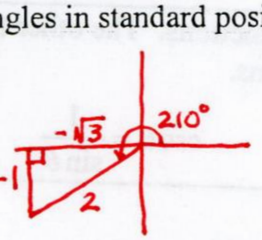
Circular function trigonometry makes use of reference angles in triangles and is really not much different than right triangle trigonometry. Think of it as an extension of right triangle trig.

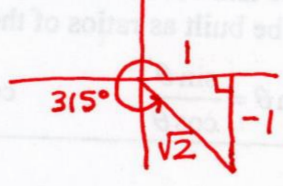
30° - 60° - 90° and 45° - 45° - 90° reference triangles can be used to find trig ratios of angles which are multiples of 30° or 45°.



Example 3: Draw angles in standard position and make "reference triangles" to find:

a. $\cos 210^\circ = -\frac{\sqrt{3}}{2}$ b. $\tan 315^\circ = -1$





Example 4: Since 2π radians $= 360^\circ$, it follows that $\pi^R = 180^\circ$, and the following common radian measures should be easy to think about in degrees. Convert each common radian measure to degrees.

a. $\frac{\pi}{2} = 90^\circ$ b. $\frac{\pi}{4} = 45^\circ$ c. $\frac{\pi}{3} = 60^\circ$ d. $\frac{\pi}{6} = 30^\circ$

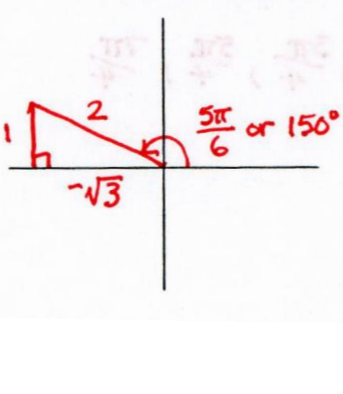
Example 5: Convert from radians to degrees or degrees to radians without using a calculator.

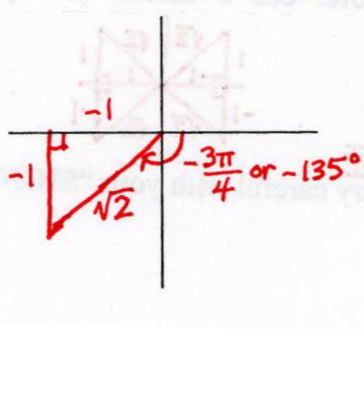
a. $\frac{5\pi}{4} = 5\left(\frac{\pi}{4}\right) = 5(45^\circ) = 225^\circ$ b. $270^\circ = 3(90^\circ) = \frac{3\pi}{2}$ c. $-120^\circ = -2(60^\circ) = -\frac{2\pi}{3}$

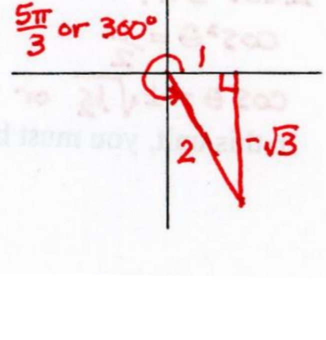
or $\frac{5\pi}{4} \cdot \frac{180^\circ}{\pi} = 225^\circ$ or $270^\circ \cdot \frac{\pi}{180^\circ} = \frac{3\pi}{2}$ or $-120^\circ \cdot \frac{\pi}{180^\circ} = -\frac{2\pi}{3}$

Examples: Draw angles in standard position, and make "reference triangles" to find the following without using a calculator:

6. $\tan\frac{5\pi}{6} = \frac{1}{-\sqrt{3}}$ 7. $\cos\left(\frac{-3\pi}{4}\right) = \frac{-1}{\sqrt{2}}$ 8. $\csc\frac{5\pi}{3} = \frac{2}{-\sqrt{3}}$

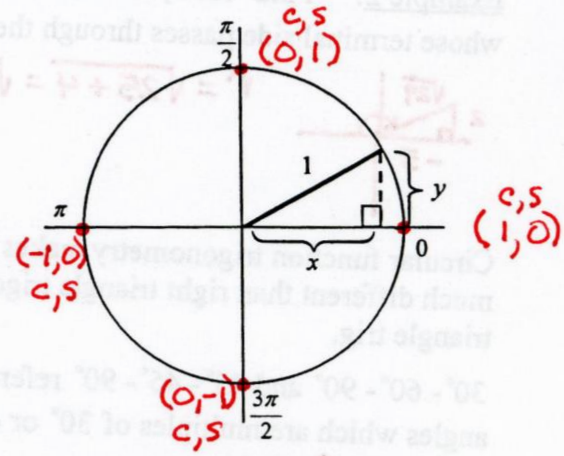






A unit circle is created by letting $r = 1$ when dealing with the circular trig functions.

Then, $\sin\theta = y$, $\cos\theta = x$, and $\tan\theta = \frac{y}{x}$.



Example 9: Use a unit circle to find:

a. $\sin\frac{\pi}{6} = \frac{1}{2}$ b. $\sin 0 = 0$ c. $\cos 0 = 1$

d. $\sin\frac{\pi}{2} = 1$ e. $\cos\frac{\pi}{2} = 0$ f. $\sin\pi = 0$

g. $\tan\pi = 0$ h. $\sin\frac{3\pi}{2} = -1$ i. $\cos\frac{3\pi}{2} = 0$

j. $\cos(-\pi) = -1$ k. $\tan\left(\frac{-\pi}{2}\right)$ = undefined

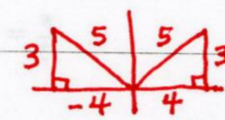
As you can see from Example 9, the unit circle is particularly useful when finding trig ratios for the quadrant separators (since no "reference triangles" can be built for them).

Sine and cosine are the two most important trig functions. The other trig functions can all be built as ratios of the sine and cosine functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

Example 10: If $\sin\theta = \frac{3}{5}$, find the possible values for

a. $\csc\theta = \frac{5}{3}$ b. $\cos\theta = \pm\frac{4}{5}$ c. $\tan\theta = \pm\frac{3}{4}$



Solving trigonometric equations requires you to "**work backwards**" from ratios to angles.

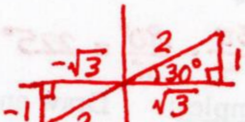
Example 11: Solve the following trig equations without using a calculator. Find all of the solutions in the interval $[0, 2\pi)$.

a. $\csc x = \frac{-2}{\sqrt{3}}$

$x = \frac{4\pi}{3}, \frac{5\pi}{3}$

b. $\cot\theta = \sqrt{3}$

$\theta = \frac{\pi}{6}, \frac{7\pi}{6}$



c. $2\cos^2\theta - 1 = 0$ **Note**: $\cos^2\theta$ means $(\cos\theta)^2$. This is trig symbolism.

$2\cos^2\theta = 1$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = \pm\sqrt{\frac{1}{2}}$ or $\pm\frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

In this unit, you must be very careful with your "**SIGNS.**"

LESSON 0.5 TRIGONOMETRY WITH A CALCULATOR, GRAPHS OF TRIGONOMETRIC FUNCTIONS

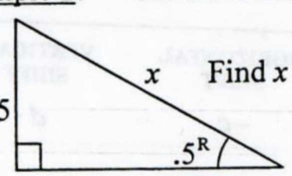
When using a calculator with trig functions, it is important that the calculator is set in the correct mode (radians or degrees). In Calculus, we will deal almost entirely with radian measure. You **will** set your calculator to radian mode prior to taking the AP exam. You will also express all calculator answers to 3 or more decimal place accuracy (unless the problem specifically asks for something else).

Example 1: Use a calculator to find:

a. $\sin 2 = .909$ b. $\tan\left(\frac{-\pi}{5}\right) = -.726$ (or $-.727$) c. $\sec 1.3 = 3.738$

Example 2: Use a calculator to find the missing measure in each triangle.

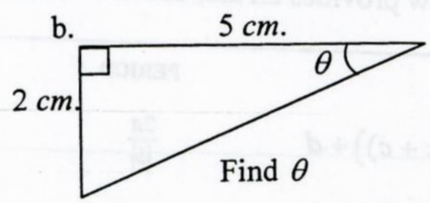
a.



$\sin .5 = \frac{5}{x}$

$x = \frac{5}{\sin .5} = 10.429$

b.



$\tan\theta = \frac{2}{5}$

$\theta = \tan^{-1}\frac{2}{5} = .380$ or $.381$

Graphs of Trig Functions:

Coterminal angles are angles having the same terminal side if placed in standard position. The fact that coterminal angles have the same trig ratios should lead you to believe that the graphs of trig functions would "repeat" every 2π radians (measured on the x-axis). They do. Actually, tangent and cotangent graphs repeat more often (every π radians).

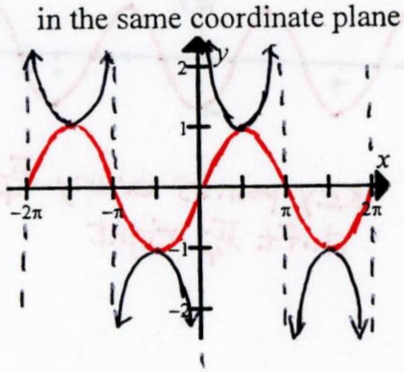
Trig functions are periodic (their graphs repeat after a certain period or cycle).
The sine, cosine, cosecant, and secant functions all have a period of 2π.
The tangent and cotangent functions have a period of π.

You should be able to easily graph the trig functions by using trig values at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and by using the fact that the functions are periodic.

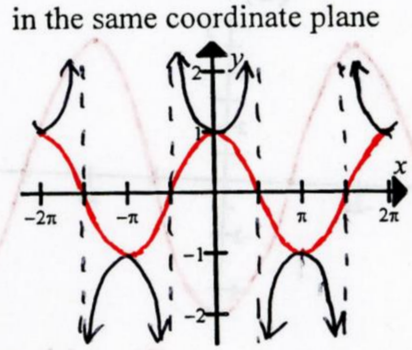
(You should also use $x = \pm\frac{\pi}{4}$ for the tangent and cotangent graphs.)

Example 3: Graph each of the following.

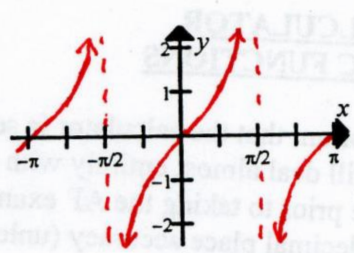
a. $y = \sin x$ and $y = \csc x$ in the same coordinate plane



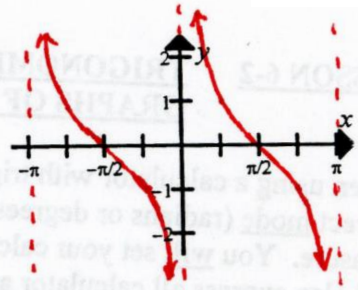
b. $y = \cos x$ and $y = \sec x$ in the same coordinate plane



c. $y = \tan x$



d. $y = \cot x$



Remember: Each of these 2 functions has a period of π.

You should be able to use the parent trig graphs to graph functions of the form $y = a\sin(b(x+c)) + d$. (sin could be replaced by any other trig function.)

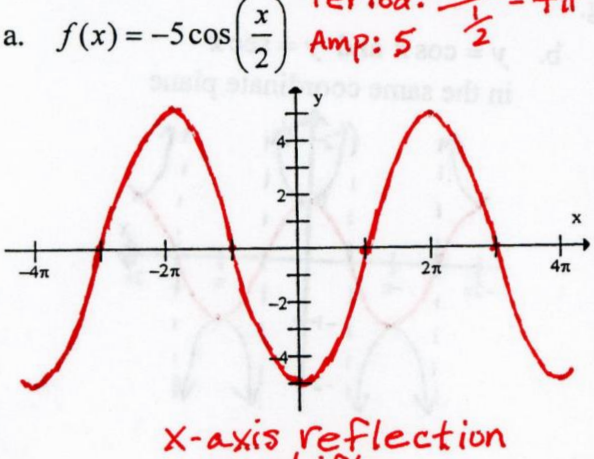
The chart below provides an aid, but remember to think of "adjustments to graphs."

FUNCTION	PERIOD	AMPLITUDE	HORIZONTAL SHIFT	VERTICAL SHIFT
$y = a\sin(b(x+c)) + d$ or $y = a\cos(b(x+c)) + d$	$\frac{2\pi}{\lvert b\rvert}$	$\lvert a\rvert$	$-c$	d
$y = a\tan(b(x+c)) + d$ or $y = a\cot(b(x+c)) + d$	$\frac{\pi}{\lvert b\rvert}$	None	$-c$	d
$y = a\sec(b(x+c)) + d$ or $y = a\csc(b(x+c)) + d$	$\frac{2\pi}{\lvert b\rvert}$	None	$-c$	d

When c is positive, the horizontal shift is to the left. When c is negative, the horizontal shift is to the right. Horizontal shift is often called phase shift for periodic functions.

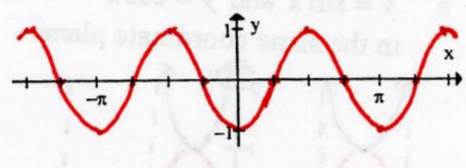
Example 4: Without using a calculator, sketch two cycles of:

a. $f(x) = -5\cos\left(\frac{x}{2}\right)$ Period: $\frac{2\pi}{\frac{1}{2}} = 4\pi$ Amp: 5



x-axis reflection
no shifts

b. $g(t) = \sin\left(2t - \frac{\pi}{2}\right) = \sin\left(2\left(t - \frac{\pi}{4}\right)\right)$ Period $= \frac{2\pi}{2} = \pi$ Amp $= 1$



Key points every $\frac{\pi}{4}$
shift $\frac{\pi}{4}$ right

The sine and cosine functions are related to each other by the basic Pythagorean Identity:

$$\begin{cases} \sin^2\theta + \cos^2\theta = 1 \\ \text{or } \sin^2 x + \cos^2 x = 1 \end{cases}$$

Example 5: Use the Pythagorean Identity to rewrite $2\cos\theta - \sin^2\theta = -2$ in a form which only contains one trig function. Then, without using a calculator, solve for θ on the interval $[0, 2\pi)$.

$2\cos\theta - (1 - \cos^2\theta) = -2$

$2\cos\theta - 1 + \cos^2\theta = -2$

$\cos^2\theta + 2\cos\theta + 1 = 0$

$(\cos\theta + 1)^2 = 0$

$\cos\theta = -1$

$\theta = \pi$

Use your calculator to verify your solution.

LESSON 0.6 EXPONENTIAL FUNCTIONS

An **exponential function** is a function represented by a constant base with a variable exponent. For example, $f(x) = 2^x$, $y = e^x$, and $g(x) = 3^{x^2-5}$ are exponential functions. These basic properties of exponents are used when working with exponential functions.

For a and b positive real numbers and x and y any real numbers:

1. $a^0 = 1$	2. $a^x a^y = a^{x+y}$	3. $\dfrac{a^x}{a^y} = a^{x-y}$
4. $\left(a^x\right)^y = a^{xy}$	5. $(ab)^x = a^x b^x$	6. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
7. $a^{-x} = \dfrac{1}{a^x}$	Note: $(a+b)^x \neq a^x + b^x$	

When simplifying, do not leave answers with negative exponents.

Examples: Simplify without using a calculator.

1. $27^{\frac{4}{3}}$

2. $\left(e + \dfrac{1}{e}\right)^0 = 1$

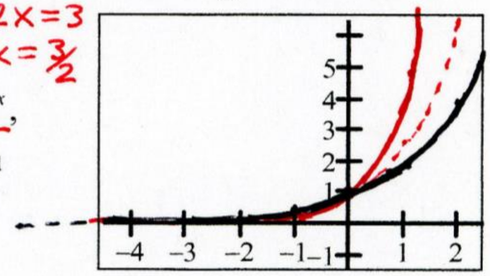
3. $\left(\dfrac{e^5 \cdot e^{-3}}{e^4}\right)^2 = \left(\dfrac{1}{e^2}\right)^2 = \dfrac{1}{e^4}$

Note: $27^{\frac{4}{3}} = \sqrt[3]{27^4} = \left(\sqrt[3]{27}\right)^4 = 3^4 = 81$

4. $5^3 \cdot 25^{-2} = 5^3 (5^2)^{-2}$
$= 5^3 \cdot 5^{-4} = 5^{-1} = \dfrac{1}{5}$

5. Solve $9^x = 27$ without using a calculator.
$(3^2)^x = 3^3$ $\quad 2x = 3$
$3^{2x} = 3^3$ $\quad x = \dfrac{3}{2}$

6. Use a calculator to carefully graph $y = 2^x$, $y = 5^x$, and $y = e^x$ in the same coordinate plane. Do you see any similarities in the graphs?



Graphs of Exponential Functions: If $f(x) = a^x$ and $a > 1$, then

1. The domain of $f(x)$ is $(-\infty, \infty)$.
   The range of $f(x)$ is $(0, \infty)$.

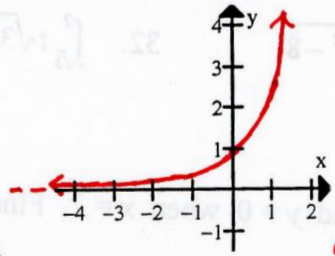
2. The graph of $f(x)$ is continuous, increasing, concave upward, and one-to-one (has an inverse function).

3. The x-axis is a horizontal asymptote to the left: $\lim_{x\to-\infty} f(x) = 0$
   (Also, $\lim_{x\to\infty} f(x) = \infty$)

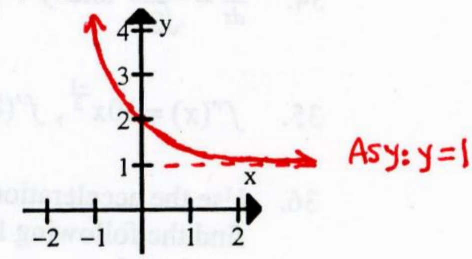
4. The y-intercept is $(0, 1)$.
   Another key point is $(1, a)$.

The letter e used as a base in Examples 2, 3, and 6, is not an unknown. It is a number called the natural base for exponential functions. It is the most common base in Calculus, because functions with base e are easier to differentiate and integrate than functions with other bases. By definition, $e = \lim_{x \to 0}(1+x)^{\frac{1}{x}}$. To three decimal places, $e \approx 2.718$.

Example 7: Without using a calculator, sketch a graph of $y = e^x$.

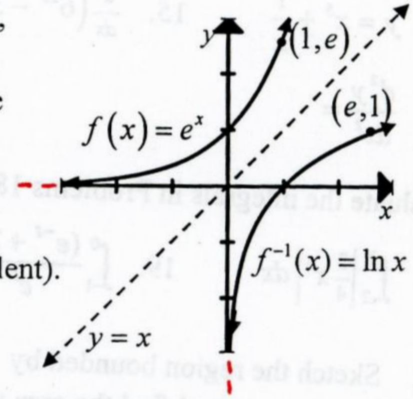


Example 8: Using adjustments to the graph from Example 7, graph $f(x) = e^{-x} + 1$ without using a calculator. Write an equation for the graph's asymptote.



LESSON 0.7 LOGARITHMIC FUNCTIONS

Since $f(x)=e^x$ is one-to-one (continuous and increasing), it must have an inverse. However, if you switch x and y in the equation $y=e^x$ to get $x=e^y$, you cannot isolate the new y by using algebraic methods. So, we must define $f^{-1}(x)$ for the function $f(x)=e^x$. For $f(x)=e^x$, $f^{-1}(x)$ is called the **natural logarithmic function**, and we write $f^{-1}(x)=\ln x$ (so that $x=e^y$ and $y=\ln x$ must be equivalent).



In general, if $f(x)=a^x$ $(a>0)$, then $f^{-1}(x)=\log_a x$ (so that $x=a^y$ and $y=\log_a x$ must be equivalent).

Note: $\log_e x$ is usually written as $\ln x$ and $\log_{10} x$ is usually written simply as $\log x$.

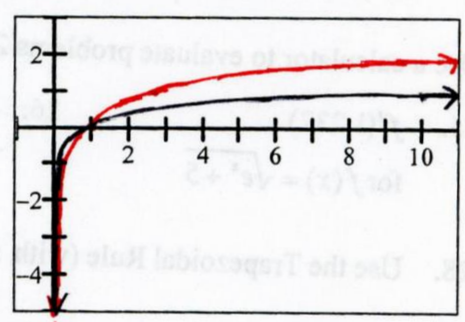
Graphs of Logarithmic Functions: If $f(x)=\log_a x$ and $a>1$, then

1. The domain of $f(x)$ is $(0,\infty)$. The range of $f(x)$ is $(-\infty,\infty)$.
2. The graph of $f(x)$ is continuous, increasing, concave downward, and one-to-one (has an inverse function).
3. The y-axis is a vertical asymptote downward: $\lim_{x\to 0} f(x)=-\infty$ (Also, $\lim_{x\to\infty} f(x)=\infty$)
4. The x-intercept is $(1, 0)$. Another key point is $(a, 1)$.

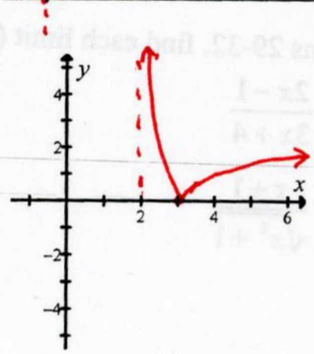
Compare these graphical characteristics of $f(x)=\log_a x$ to those of $f(x)=a^x$ from Lesson 5-1 (page 131).

Example 1: Use a calculator to graph $y=\ln x$ and $y=\log x$ in the same coordinate plane. Do you see any similarities in the graphs?

Satisfy characteristics in box above.



Example 2: Without using a calculator, sketch a graph of $y=|\ln(x-2)|$. Write an equation for the graph's asymptote.



For changing forms of an equation involving exponentials or logarithms, we use the following **Change of Form Definition**:

Exponential form	$x = e^y \leftrightarrow y = \ln x$ $x = a^y \leftrightarrow y = \log_a x$	Logarithmic form

Example 3: Change the following equations from exponential form to logarithmic form or vice versa.

a. $3^4 = 81$ $\log_3 81 = 4$ b. $e^0 = 1$ $\ln 1 = 0$ c. $\log(.1) = -1$ $10^{-1} = .1$

Example 4:

a. Since $e^0 = 1$, $\ln 1 = 0$ b. Since $e^1 = e$, $\ln e = 1$

c. Because the natural exponential function and the natural logarithmic function are inverses, $\ln e^n = e^{\ln n} = n$

Example 5: Use the inverse idea from Example 4c. to simplify.

a. $\ln e^{\sqrt{2}} = \sqrt{2}$ b. $e^{\ln(3x)} = 3x$ c. $10^{\log 2} = 2$ d. $\log_2 2^{x^2} = x^2$

Properties of Logarithms:

1. $\ln(ab) = \ln a + \ln b$
2. $\ln \frac{a}{b} = \ln a - \ln b$
3. $\ln a^n = n \ln a$

These properties work for any bases, but only if $a > 0$ and $b > 0$

Example 6: Expand using Logarithm Properties 1-3 above.

a. $\ln \frac{5}{8} = \ln 5 - \ln 8$

b. $\ln \sqrt[3]{x^2+1} = \ln(x^2+1)^{\frac{1}{3}} = \frac{1}{3}\ln(x^2+1)$

Example 7: Condense into a single logarithm. ($x > 0$ and $y > 0$)

a. $-3\ln x + 5\ln y$

$\ln x^{-3} + \ln y^5$

$= \ln(x^{-3} \cdot y^5)$ or $\ln\left(\frac{y^5}{x^3}\right)$

b. $\frac{1}{2}\ln x + \ln(x+1) - 3\ln y$

$\ln x^{\frac{1}{2}} + \ln(x+1) - \ln y^3$

$= \ln \frac{x^{\frac{1}{2}}(x+1)}{y^3}$

Example 8: Solve for x.

a. $y = e^{2x-5} + 6$

$y - 6 = e^{2x-5}$

$\ln(y-6) = 2x - 5$

$5 + \ln(y-6) = 2x$

$\frac{5 + \ln(y-6)}{2} = x$

b. $\log_2 x - \log_2(x-8) = 3$

$\log_2 \frac{x}{x-8} = 3$

$2^3 = \frac{x}{x-8}$

$8 = \frac{x}{x-8}$

$8x - 64 = x$

$7x = 64$

$x = \frac{64}{7}$

Checks out

Change of Base Formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$

Since the only two logarithmic bases on your calculator are 10 (log key) and e (ln key), you will change bases on your calculator in one of two ways:

$$\log_a x = \frac{\log x}{\log a} \qquad \text{or} \qquad \log_a x = \frac{\ln x}{\ln a}$$

Example 9: Use your calculator to find $\log_7 112$ to 3 or more decimal places. $\log_7 112 = \dfrac{\log 112}{\log 7}$ or $\dfrac{\ln 112}{\ln 7} = 2.424$

Example 10:

a. Find an exact value for x, if $3^{x+2} = 6$. $\log_3 6 = x+2$

$x = -2 + \log_3 6$

b. Use your calculator to find a decimal value for your answer from Part a. to 3 or more decimal places. $x = -.369$

AP CALCULUS AB **Summer Assignment: Do ALL Problems on another piece of paper.**

Lesson 0.1

Draw accurate graphs for the following without using a calculator.

1. $4x+2y=6$ 2. $y=\dfrac{-x+4}{2}$

3. Find equations for lines passing through $(-1,3)$ with the following characteristics.

   a. $m=\frac{2}{3}$ b. parallel to $2x+4y=7$

   c. passing through the origin d. perpendicular to the x-axis

Use a calculator for Problems 4-6. Remember to show three or more decimal place accuracy for all answers that are not exact.

4. Solve $3x^3-3x+1\le 0$.
5. Solve $|3x+5|>2$.
6. Find the x-value(s) of the point(s) of intersection for the graphs of $x-y^2=-7$ and $2x-3y+12=0$. Write the equation you are solving.

Lesson 0.2

Draw accurate graphs for the following without using a calculator. Use the parent graphs on Page 6 to help you whenever possible.

7. $y=\dfrac{1}{x}+1$ 8. $y=\sqrt[3]{x-2}$ 9. $y=|x^2-2|$ 10. $y=x^{\frac{2}{3}}-1$

11. $y-x^2=0$ 12. $x=y^2$ 13. $y=x^3-1$

14. For which of the relations in Problems 7-13 is y not a function of x?

15. If $f(x)=1-x^3$ and $g(x)=2x+1$, find the following.

   a. $f(x)+g(x)$ b. $f(g(x))$ c. $(g\circ f)(2)$

Find the zeros without using a calculator.

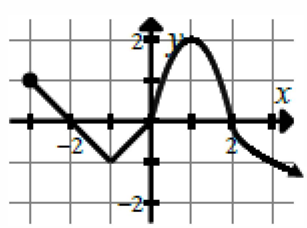
16. $f(x)=x^4-7x^2+12$ 17. $g(x)=\begin{cases}2x-1, & x<0\\ x^2-4, & x\ge 0\end{cases}$

18. Find the inverse function for $f(x)=(x^3-1)^5$.

Use the graph of $y=f(x)$ at the right to draw an accurate graph for each of the following.

19. $y=\frac{1}{2}|f(x)|$ 20. $y=|f(2x)|$

21. $y=f(x-2)-2$



22. Without using a calculator sketch a graph of $g(x)=\begin{cases} x-1, & x\le 0 \\ x^2-1, & 0<x<2 \\ 4, & x\ge 2 \end{cases}$

Lesson 0.3

Without using a calculator, find the point(s) of intersection of the graphs of the following.
Show algebra steps!

23. $\begin{cases} y=x+5 \\ y=-2x+8 \end{cases}$

24. $\begin{cases} x^2-y^2=9 \\ x^2+y^2=9 \end{cases}$

For each function in Problems 25-27, without using a calculator:

a. find the domain and the range.
b. find the intercepts.
c. discuss the symmetry.
d. tell whether the function is even, odd, or neither.
e. draw an accurate graph.

25. $y=\left|x^2-4\right|$

26. $y=-\sqrt{x+4}$

27. $y=\left|x^3\right|-2$

Use a calculator for Problems 28-30. Remember to show three or more decimal place accuracy for all answers that are not exact.

a. find the domain and the range.
b. find the intercepts.
c. discuss the symmetry.
d. tell whether the function is even, odd, or neither.
e. draw an accurate graph.

28. $y=\dfrac{x}{x^2-4}$

29. $y=-\sqrt{5-2x^2}$

30. $y=3x^2-3x-5$

Are the following functions even, odd, or neither? Do not use a calculator.

31. $g(x)=\dfrac{x}{\left|x^3-x\right|}$

32. $h(x)=\dfrac{x-1}{\left|x^3-x\right|}$

Lesson 0.4

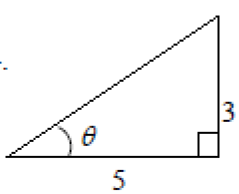
33. Without a calculator, convert from degrees to radians for Part a. and radians to degrees for Part b.

a. 300°

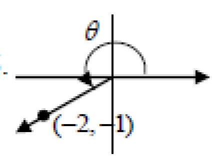
b. $-\frac{5\pi}{2}$

For Problems 34-36, find $\sin\theta$, $\cos\theta$, and $\tan\theta$ without using a calculator.

34.

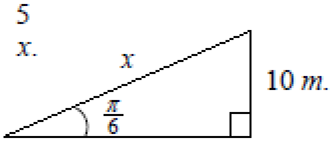


35.



36. The measure of θ is $\frac{8\pi}{3}$.

37. Find x.



Evaluate each of the following without using a calculator.

38. $\sin\pi$ 39. $\cos\frac{3\pi}{2}$ 40. $\tan\frac{4\pi}{3}$ 41. $\sec\frac{5\pi}{4}$ 42. $\csc\frac{7\pi}{6}$ 43. $\cot\frac{-3\pi}{2}$

44. Given $\tan\theta=\frac{-5}{12}$, $\sin\theta>0$, find $\cos\theta$.

Solve the following equations on the interval $\left[0,2\pi\right)$.

45. $3\tan^2\theta-1=0$ 46. $3\csc^2\theta-4=0$

Lesson 0.5

47. Use a calculator to solve $\tan^2 x=\sin(2x)+3$ on $[0,\pi)$.

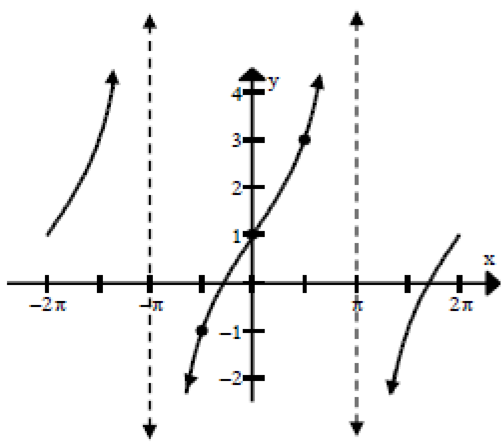
Do not use a calculator for Problems 48-50.

48. Find the amplitude, period, and phase shift for $y=\frac{-3}{4}\cos(3x-\pi)$.

49. Sketch the graph of $y=-\frac{1}{2}\sin\left(x-\frac{\pi}{2}\right)$.

50. Find the discontinuities of $f(x)=\csc(4x)$.

51. Write an equation of the form $y=a\tan\left(b(x-c)\right)+d$ for the graph below.



Lesson 0.6 and 0.7

For Problems 52 and 53, find the inverse of the given function, and then sketch the function and its inverse in the same coordinate plane.

52. $f(x) = \ln(-x)$

53. $g(x) = e^{2x} + 1$

Simplify the expressions in Problems 54 and 55 without using a calculator.

54. $\log_3 \frac{1}{27}$

55. $\ln \frac{e^{10}}{e^3}$

56. Use Log Properties to expand $\ln\left(x\sqrt{y-1}\right)$.

57. Use Log Properties to condense $2\log p - 3\log q - \log r$ into a single logarithm.

Solve for x without a calculator:

58. $e^{2x-3} - 5 = 0$

59. $1 - 3\ln x = -5$

60. Without using a calculator, list the domain, two points that the graph of the function contains, and the asymptote for the graph of the function. Then sketch each graph in the same coordinate plane.

a. $y = e^x$

b. $y = \ln x$

61. Using adjustments to the graph of $y = e^x$ from Problem 60, sketch $y = e^{-x} - 3$. What is the asymptote for this graph?

62. Find the inverse of $y = e^{-x} - 3$, and sketch its graph in the same coordinate plane that you used for Problem 61. What is the asymptote for this graph?

For Problems 63-67, solve for x without using a calculator. Simplify your answers.

63. $\log_5 x = -2$

64. $e^{\ln(-2x+3)} = 5$

65. $\ln|2x-1| = 0$

66. $\log_3 x = \log_3(2x+1) - \log_3(x+4) + 1$

67. $4^3 = 8^{2x-1}$

UNIT 0 SUMMARY

Slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Equations for lines:

Point-Slope form $y - y_1 = m(x - x_1)$

Slope-Intercept form $y = mx + b$ (where b is the y-intercept)

Domain: all possible x-values

Range: all possible y-values

Inverse functions: found by switching x and y and solving for the new y.

Parent graphs and graphing adjustments: see Page 6

Intercepts:

To find the x-intercept, let $y = 0$ and solve for x.

To find the y-intercept, let $x = 0$ and solve for y.

Symmetry:

Informal tests:

1. y-axis: substituting a number and its opposite for x give the same y-value.
2. x-axis: substituting a number and its opposite for y give the same x-value.
3. origin: substituting a number and its opposite for x give opposite y-values.

Even/odd functions:

Even functions have graphs with y-axis symmetry.

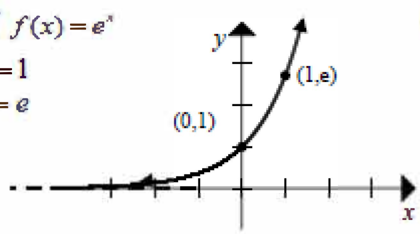
Odd functions have graphs with origin symmetry.

Exponential and Logarithmic Graphs:

Graph of $f(x) = e^x$

$e^0 = 1$

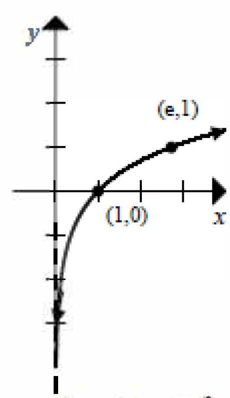
$e^1 = e$



Graph of $g(x) = \ln x$

$\ln 1 = 0$

$\ln e = 1$



All basic exponential $\left(f(x) = a^x\right)$ and logarithmic $\left(g(x) = \log_a x\right)$ graphs with $a > 0$ are similar to the graphs shown above.

$f(x) = e^x$ and $g(x) = \ln x$ are inverse functions, so $\ln e^x = e^{\ln x} = x$.

Change of Form Definition:

Exponential form $\left\{\begin{array}{l} x = e^y \leftrightarrow y = \ln x \\ x = a^y \leftrightarrow y = \log_a x \end{array}\right\}$ Logarithmic form

Properties of Logarithms: only true when $a > 0$ and $b > 0$

1. $\ln(ab) = \ln a + \ln b$
2. $\ln \frac{a}{b} = \ln a - \ln b$
3. $\ln a^n = n \ln a$

Change of Base: $\log_a x = \frac{\ln x}{\ln a}$